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Cascade with Subsonic Leading-Edge Locus

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Introduction

In modern aircraft engine technology there is considerable interest in the problem of the unsteady supersonic cascade with subsonic axial velocity. In this Note, we consider a two-dimensional oscillating cascade with a subsonic leading-edge locus in a supersonic flow which is uniform far upstream. It is assumed that the blades have small thickness and camber and are undergoing small amplitude harmonic oscillations. Kurosaka[1] has obtained a low frequency analytical solution to this problem and Verdon[2] has obtained a finite-difference solution. This Note will show that the problem can be reduced solving a functional integral equation and give a representation of the kernel function which is useful for computation. The detailed structure of this equation is explicitly displayed.

Derivation of Integral Equation

The blades, which are assumed to have small thickness and camber, are undergoing small amplitude harmonic oscillations. We suppose that all lengths are nondimensionalized by the half-blade chord $c/2$; the time t is nondimensionalized with respect to $c/2$ divided by the freestream velocity U; the pressure fluctuation p is nondimensionalized by ρ_0, the freestream density, times U^2; and the upwash velocity v is nondimensionalized by U. Let a denote the freestream speed of sound. Then the pressure fluctuation is governed by the equation (see Fig. 1)

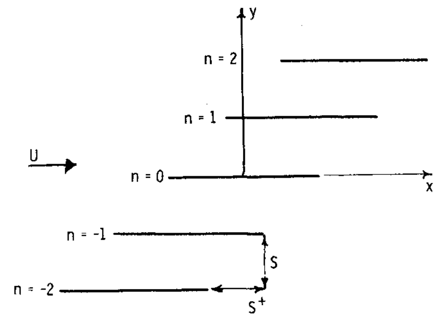

Fig. 1 Dimensionless cascade configuration.

$$\frac{\partial^2\psi}{\partial y^2} - \beta^2\frac{\partial^2\psi}{\partial x^2} - \beta^2 k^2\psi = 0 \tag{1}$$

where

$$\psi \equiv p e^{i(\omega t - Mkx)} \tag{2}$$

$\beta^2 = M^2 - 1$; $k = \omega M/\beta^2$; $M = U/a$ is the freestream Mach number; and the upwash velocity, v, related to the pressure by

$$e^{i\beta^2 kx/M}\frac{\partial}{\partial x}(e^{-i\beta^2 kx/M}V) = -e^{iMkx}\frac{\partial\psi}{\partial y} \tag{3}$$

where $V = v\exp[i\omega t]$.

The upwash velocity on the nth blade is assumed to differ from that of the 0th blade by only a phase factor so that

$$V(x+ns^\dagger, ns) = e^{in\sigma}V(x,0) \quad \text{for } |x|<1, n=0,\pm 1,\pm 2 \tag{4}$$

where σ is the interblade phase angle and the upwash velocity on the 0th blade is related to its displacement $W_0 e^{-i\omega t}$ by

$$V(x,0) = \left[-i\omega + \frac{\partial}{\partial x}\right]W_0(x) \quad \text{for } |x|<1$$

As is usual, suppose for convenience that the frequency has a small positive imaginary part which we shall set equal to zero at the end of the analysis. Then $k = k_r + i\epsilon$ for $0<\epsilon\ll 1$ and the outgoing wave boundary condition at infinity is now replaced by a boundedness condition.

Since Eq. (1) possesses the separation of variables solution exp. $\{-[i(\alpha x - \beta\gamma y)]\}$, where

$$\gamma \equiv \sqrt{\alpha^2 - k^2}$$

the boundary condition Eq. (4) suggests that we seek a solution in the form of the super-position

$$\psi = \sum_{n=-\infty}^{\infty}\psi_n \tag{5}$$

where

$$\psi_n \equiv (\operatorname{sgn} y_n)/2\int_{-\infty+i\delta}^{\infty+i\delta} f_n(\alpha)e^{-i(\alpha x_n - \beta\gamma|y_n|)}d\alpha \tag{6}$$

we have put (see Fig. 1) $x_n = x - ns^\dagger$; $y_n = y - ns$ for $n = 0, \pm 1, \pm 2, \ldots$, and in order to insure that the solution remains boun-

Received January 24, 1975; revision received March 17, 1975.
Index categories: Nonsteady Aerodynamics; Supersonic and Hypersonic Flow; Airbreathing Propulsion, Subsonic and Supersonic.

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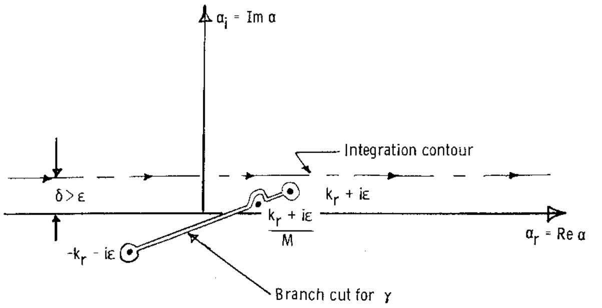

Fig. 2 Integration contour and branch cut in complex α-plane.

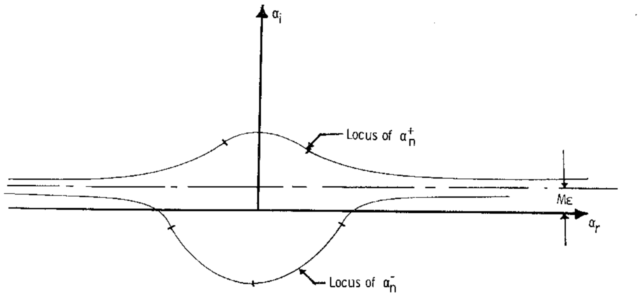

Fig. 3 Approximate locus of roots in complex α-plane.

ded at infinity we have chosen the branch cut for the square root, γ, and the integration contour, $\alpha_r + i\delta$, in the manner shown in Fig. 2 (with $\delta > \epsilon$). This solution possesses the jump discontinuity

$$[\psi(x)] = [\psi_n(x)] = \int_{-\infty+i\delta}^{\infty+i\delta} f_n(\alpha) e^{-i\alpha x_n} d\alpha \tag{7}$$

across the line $y = ns$ passing through the nth blade since it is only possible to satisfy the requirement that the upwash velocity be continuous by allowing a discontinuity in the pressure. The resulting pressure discontinuity in front of and behind the blade will be eliminated in the subsequent analysis.

Since the upwash velocity, v, vanishes at infinity, Eq. (3) can be integrated to obtain

$$V = -e^{i\beta^2 kx/M} \int_{-\infty}^{x} e^{ikx'/M} \frac{\partial \psi}{\partial y} (x', y)\, dx'$$

Inserting Eqs. (5) and (6) and carrying out the integration now yields

$$V = \frac{1}{2i} e^{iMkx} \frac{\partial}{\partial y} \int_{-\infty+i\delta}^{\infty+i\delta} \frac{Mf_0(\alpha)}{M\alpha - k} \times \sum_{n=-\infty}^{\infty} (\operatorname{sgn} y_n) f_n(\alpha) e^{-i(\alpha x_n - \gamma\beta |y_n|)} d\alpha \tag{8}$$

If we put $f_n(\alpha) = f_0(\alpha)\exp in\Gamma$ where $\Gamma = \sigma - Mks^\dagger$ it is easy to show from Eq. (8) that $V(x+ns,^\dagger\ y+ns) = e^{in\sigma} V(x,y)$. Hence the boundary condition (4) is automatically satisfied and

$$V = \frac{1}{2i} \frac{\partial}{\partial y} \int_{-\infty+i\delta}^{\infty+i\delta} \frac{Mf_0(\alpha)}{M\alpha - k} \times \sum_{n=-\infty}^{\infty} (\operatorname{sgn} y_n)\, e^{i[n\sigma - (\alpha - Mk)x_n + \beta\gamma |y_n|]}\, d\alpha \tag{9}$$

On the other hand since $[\psi(x)] = 0$ for $|x| > 1$ we can invert the Fourier transform in Eq. (7) (with $n = 0$) to obtain

$$f_0(\alpha) = \frac{1}{2\pi} \int_{-1}^{1} [\psi] e^{i\alpha x} dx \tag{10}$$

Upon inserting this into Eq. (9) and interchanging the order of integration we obtain

$$V(x,y) = \int_{-1}^{1} K(x - x', y)\ [P(x')]\ dx' \tag{11}$$

where $P \equiv pe^{i\omega t} = \psi\, e^{iMkx}$ and

$$K(x,y) \equiv \frac{M}{i4\pi} \frac{\partial}{\partial y} \int_{-\infty+i\delta}^{\infty+i\delta} \frac{\gamma}{M\alpha - k} \times \sum_{n=-\infty}^{\infty} (\operatorname{sgn} y_n)\, e^{i[n\sigma - (\alpha - Mk)x_n + \beta\gamma |y_n|]} d\alpha \tag{12}$$

By letting $y \to 0$ we obtain an integral equation for the pressure jump $[P]$ across the 0th blade in terms of the known upwash velocity on the blade surface. viz

$$V(x,0) = \int_{-1}^{1} K_0(x - x')\ [P(x')]\ dx' \tag{13}$$

where

$$K_0(x) \equiv \lim_{y \to 0} K(x,y) \tag{14}$$

Expression for the Kernel Function

The forms (12) and (14) for the Kernel function are not suitable for numerical evaluation, because the integral will not converge if we just put $\delta = \epsilon = 0$ in the integrand. To carry out this limit, it is convenient to express the kernel in a different form. To this end notice that since $\mathcal{I}m(\alpha - Mk\delta) = 0$ and $\mathcal{I}m\gamma > 0$ for $\delta = M\epsilon$ $(-\infty < \alpha_r < \infty)$, it follows that $|\exp[i(\alpha - Mk)ns^\dagger + \beta\gamma |y_n|]| < 1$, and we can use the geometric series

$$\sum_{n=0}^{\infty} z^n = (1 - z)^{-1}$$

to evaluate the infinite series which appears in the integrand of Eq. (12) to obtain for $0 < y < s$

$$K_0(x) = -\frac{1}{8\pi} \lim_{y \to 0} \frac{\partial}{\partial y} \int_{-\infty+i\epsilon M}^{\infty+i\epsilon M} \frac{e^{-i(\alpha - Mk)x}}{(\alpha - k/M)} \times \left[\frac{e^{i(\Delta_- + \beta\gamma y)}}{\sin \Delta_-} + \frac{e^{i(\Delta_+ - \beta\gamma y)}}{\sin \Delta_+} \right] d\alpha \tag{15}$$

where

$$\Delta_\pm \equiv \tfrac{1}{2}[\sigma - Mks^\dagger + \alpha s^\dagger \pm \beta\gamma s] \tag{16}$$

At first glance it might appear that the integrand in this expression possesses branch points due to the appearance of the radical γ. However, it can easily be verified by replacing γ by $-\gamma$ that this function depends only on γ^2 so that the branch points are therefore "cancelled" and the integrand possesses only poles. We can therefore use Jordan's lemma to evaluate the integral in terms of its residues. To this end, notice that the poles of the integrand occur at $\alpha = k/M$ and at the points where $\Delta_\pm = n\pi$ for $n = 0, \pm 1, \pm 2\ldots$, But it follows from Eq. (16) that the latter points are determined by

$$\alpha_n^{\pm} = \Gamma_n \frac{s^\dagger}{d^{\dagger 2}} \pm \frac{s\beta}{d^\dagger} \left[\left(\frac{\Gamma_n}{d^\dagger} \right)^2 - k^2 \right]^{1/2} \tag{17}$$

where we have put $\Gamma_n \equiv -\sigma + Mks^\dagger + 2n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$ and $d^\dagger \equiv (s^{\dagger 2} - \beta^2 s^2)^{1/2}$. Notice that $d^\dagger$ is always real for a subsonic leading-edge locus. The + sign corresponds to the roots which lie in the upper half plane, and the minus sign to those in the lower half plane. The locus of roots in the complex α-plane is shown in Fig. 3.

When $x<0$ we must close the contour in the upper half plane and when $x>0$ in the lower half plane. Hence

$$K_0(x)=\begin{cases}K^+(x) & x<0\\ K^-(x) & x>0\end{cases} \quad (18)$$

where $K^\pm(x)\equiv\pm\sum \text{Res in} \binom{\text{upper}}{\text{lower}}$ half plane.

Then upon evaluating the residues we find that

$$K^+(x)=\frac{1}{2i}\lim_{y\to 0}\frac{\partial}{\partial y}$$
$$\times\sum_{n=-\infty}^{\infty}\frac{(\Gamma_n-\alpha_n^+s^\dagger)e^{-i[(\alpha_n^+-Mk)x+(\Gamma_n-\alpha_n^+s^\dagger)y/s]}}{(\alpha_n^+-k/M)\;(s^\dagger\Gamma_n-d^{\dagger 2}\alpha_n^+)} \quad (19)$$

and

$$K^-(x)=\frac{\omega}{2}\;\frac{\sinh(\omega s)\;e^{i\omega x}}{\cosh(\omega s)-\cos(\sigma-s\dagger\omega)}$$
$$-\frac{1}{2i}\lim_{y\to 0}\frac{\partial}{\partial y}$$
$$\times\sum_{n=-\infty}^{\infty}\frac{(\Gamma_n-\alpha_n^-s^\dagger)\;e^{-i[(\alpha_n^--Mk)x+(\Gamma_n-\alpha_n^-s^\dagger)y/s]}}{(\alpha_n^--k/M)\;(s^\dagger\Gamma_n-d^{\dagger 2}\alpha_n^-)} \quad (20)$$

These series are only conditionally convergent and will not converge at all if we take the derivative term by term. To obtain convergent series, notice that since $\alpha_n^\pm\sim\Gamma_n/(s^\dagger\pm\beta s)+0(n^{-1})$ as $n\to\infty$ the nth term of these sums behave like

$$\left\{\exp\left[-i\Gamma_n\left[\frac{x\pm\beta y}{s^\dagger\pm\beta s}\right]\right]\right\}/\Gamma_n$$

The series composed of these terms will converge to a row of step functions. Hence, its derivative will converge to a row of delta functions. We can evaluate the latter series by using the theory of distributions to show that[3]

$$\lim_{y\to 0}\frac{\partial}{\partial y}\sum_{n=-\infty}^{\infty}\frac{1}{\Gamma_n}e^{-i\Gamma_n(x\mp\beta y)/(s^\dagger\pm\beta s)}$$
$$=\pm\frac{i\beta}{s^\dagger\pm\beta s}e^{i[(\sigma-Mks^\dagger)/(s^\dagger\pm\beta s)]x}\sum_{n=-\infty}^{\infty}e^{-(2in\pi x)/(s^\dagger\pm\beta s)}$$
$$=\pm i\beta\sum_{n=-\infty}^{\infty}e^{in(\sigma-Mks^\dagger)}\delta(x_n\pm\beta sn)$$

Hence

$$K^\pm(x)=\tilde{K}^\pm(x)+\frac{\beta}{2}\sum_{n=-\infty}^{\infty}e^{i(n\sigma+Mkx_n)}\delta(x_n\pm\beta sn) \quad (21)$$

where

$$\tilde{K}^+\equiv-\frac{e^{iMkx}}{2s}\sum_{n=-\infty}^{\infty}\left[\frac{(\Gamma_n-\alpha_n^+s^\dagger)^2e^{-i\alpha_n^+x}}{(\alpha_n^+-k/M)\,(s^\dagger\Gamma_n-d^{\dagger 2}\alpha_n^+)}+\frac{s\beta e^{-i\Gamma_nx/(s^\dagger-\beta s)}}{s^\dagger-\beta s}\right]$$

and

$$\tilde{K}^-\equiv\frac{\omega\sinh(\omega s)\;e^{i\omega x}}{2[\cosh(\omega s)-\cos(\sigma-s^\dagger\omega)]}$$
$$+\frac{e^{iMkx}}{2s}\sum_{n=-\infty}^{\infty}\left[\frac{(\Gamma_n-\alpha_n^-s^\dagger)^2e^{-i\alpha_n^-x}}{(\alpha_n^--k/M)\,(s^\dagger\Gamma_n-d^{\dagger 2}\alpha_n^-)}-\frac{s\beta e^{-i\Gamma_nx/(s^\dagger+\beta s)}}{s^\dagger+\beta s}\right]$$

are now convergent series. The kernel function is given by Eqs. (18) and (21). Only a finite number of the infinite row of delta functions in Eq. (21) will contribute to the integral in Eq. (13). However when this kernel is substituted into Eq. (13), we obtain a functional integral equation (and not an ordinary integral equation) due to the introduction of terms of the form $[P(x_n+ns\beta)]$ caused by the integration over the delta functions.

The series which appear in $\tilde{K}^\pm$ are only conditionally convergent. But the same device that was used to make the original series converge can also be used to render these latter series absolutely convergent. This removal of the slowly convergent part of the series results in a row of step functions which explicitly exhibit the discontinuities of $\tilde{K}^\pm$ (and occur at the points $x_n=\pm\beta sn$). The remaining series will represent continuous functions and will be quite suitable for numerical computation.

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Mixing Length in Low Reynolds Number Compressible Turbulent Boundary Layers

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Nomenclature

C_f = skin friction coefficient, $\tau_w/1/2\,\rho_e u_e{}^2$
ℓ = mixing length
M = Mach number
N = power law velocity exponent, Eq. (1)
R_θ = Reynolds number based on momentum thickness
T = temperature
u = velocity
u_τ = friction velocity, $(\tau/\rho)^{1/2}$
y = normal coordinate
$\rho\overline{u'v'}$ = Reynolds stress
δ = velocity boundary-layer thickness
τ = shear stress
ρ = density
γ = ratio of specific heats
δ^+ = $\delta(u_{\tau,w}\rho_w/\mu_w)$
μ = viscosity

Subscripts
m = maximum value, evaluated herein at $y/\delta=0.5$
e = edge
w = wall
t = stagnation

Received January 27, 1975; revision received March 27, 1975.
Index categories: Boundary Layers and Convective Heat Transfer—Turbulent; Nozzle and Channel Flow; Supersonic and Hypersonic Flow.

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